IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Differential calculi on commutative algebras

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1995 J. Phys. A: Math. Gen. 28 3197

(http://iopscience.iop.org/0305-4470/28/11/019)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 02/06/2010 at 00:11

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 28 (1995) 3197–3222. Printed in the UK

Differential calculi on commutative algebras

H C Baehr†, A Dimakis‡ and F Müller-Hoissen†§

† Institut für Theoretische Physik, Bunsenstrasse 9, D-37073 Göttingen, Germany
‡ Department of Mathematics, University of Crete, GR-71409 Iraklion, Crete, Greece
§ Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, D-80805 Munich, Germany*

Received 12 December 1994, in final form 15 March 1995

Abstract. A differential calculus on an associative algebra $\mathcal{A}$ is an algebraic analogue of the calculus of differential forms on a smooth manifold. It supplies $\mathcal{A}$ with a structure on which dynamics and field theory can to some extent be formulated, in very much the same way we are used to in the geometrical arena underlying classical physical theories and models. In previous work, certain differential calculi on a commutative algebra exhibited relations with lattice structures, stochastics, and parametrized quantum theories. This motivated the present systematic investigation of differential calculi on commutative and associative algebras. Various results about their structure are obtained. In particular, it is shown that there is a correspondence between first-order differential calculi on such an algebra and commutative and associative products in the space of 1-forms. An example of such a product is provided by the Itô calculus of stochastic differentials. For the case where the algebra $\mathcal{A}$ is freely generated by 'coordinates' x^i, $i = 1, \ldots, n$, we study calculi for which the differentials dx^i constitute a basis of the space of 1-forms (as a left-$\mathcal{A}$-module). These may be regarded as 'deformations' of the ordinary differential calculus on $\mathbb{R}^n$. For $n \leqslant 3$ a classification of all (orbits under the general linear group of) such calculi with 'constant structure functions' is presented. We analyse whether these calculi are reducible (i.e. a skew tensor product of lower-dimensional calculi) or whether they are the extension (as defined in this paper) of a one-dimension-lower calculus. Furthermore, generalizations to arbitrary n are obtained for all these calculi.

1. Introduction

During the last few years there has been rapidly increasing interest in 'non-commutative geometry'. Basically, this notion stands for an attempt to get away from the classical concept of a (differentiable) manifold as the arena in which physics takes place. In particular, this is strongly motivated by considerations about spacetime structure at very small length scales, and quantum gravity. The manifold is replaced by some abstract algebra $\mathcal{A}$ which is usually assumed to be associative, but not necessarily commutative. In order to be able to formulate dynamics and field theories on or with such 'generalized spaces', a convenient tool appears to be a 'differential calculus' on it, which is an algebraic analogue of the calculus of differential forms on a manifold†.

If the algebra $\mathcal{A}$ is commutative, then one can construct a (topological) space on which it can be realized as an algebra of functions. Besides the familiar continua this case also includes finite or, more generally, discrete spaces. Differential calculi on commutative

* Temporary address.

† Such a point of view has been pioneered by Robert Hermann [1].

algebras have been considered and explored in several papers (see [2–6], for example). If the algebra $\mathcal{A}$ is (freely) generated by 'coordinates' x^k, $k = 1, \ldots, n$ (together with a unit), a differential calculus on it can be specified via commutation relations with their 'differentials'

$$[\mathrm{d}x^k, x^\ell] = C^{k\ell}{}_m \, \mathrm{d}x^m \tag{1.1}$$

where $C^{k\ell}{}_m \in \mathcal{A}$ (subject to certain constraints)†. An example of interest for physics is given by

$$[\mathrm{d}x^k, x^\ell] = a^k \, \delta^{k\ell} \, \mathrm{d}x^k \qquad \text{(no summation)} \tag{1.2}$$

which may be regarded as the basic structure underlying lattice theories [3] (a^k plays the role of the lattice spacing in the kth direction). Another example is

$$[\mathrm{d}x^k, x^\ell] = \gamma \, g^{k\ell} \, \mathrm{d}x^{n+1} \qquad [\mathrm{d}x^{n+1}, x^k] = 0 \tag{1.3}$$

where $g^{k\ell}$ are the components (with respect to coordinates x^i on a manifold) of a real contravariant tensor field. For $\gamma = \mathrm{i}\hbar$ this may be viewed as a basic structure underlying parametrized (proper time) quantum theories [4]. For real and positive definite $\gamma \, g^{ij}$ one recovers the Itô calculus of stochastic differentials [5]. These examples motivate a systematic investigation of the possibilities. In [3] all differential calculi subject to (1.1) with $n = 2$ and constant structure functions, i.e. $C^{k\ell}{}_m \in \mathbb{C}$, were classified‡. The procedure used there does not extend to $n > 2$, however. We therefore propose here an alternative method and present the classification of three-dimensional calculi (see also [9]).

Section 2 recalls some basic definitions and constructions used in what follows. Section 3 presents general results about differential calculi on a commutative (and associative) algebra $\mathcal{A}$. In particular, it is shown that every (first-order) differential calculus on $\mathcal{A}$ determines an $\mathcal{A}$-bilinear commutative and associative product in the space of 1-forms. This relates the problem of classifying (first-order) differential calculi to that of determining all $\mathcal{A}$-bimodules with such product structures. This correspondence generalizes the relation established in [5] between the Itô calculus of stochastic differentials (where one has a product in the space of 1-forms) and a differential calculus of the form (1.3).

In section 4 we consider the case where $\mathcal{A}$ is freely generated (as a commutative and associative algebra) by elements x^i, $i = 1, \ldots, n$, together with a unit $\mathbf{1}$. The class of differential calculi for which the set of differentials $\mathrm{d}x^i$ are a basis of the space of 1-forms (as a left-$\mathcal{A}$-module) is then explored in some detail. They may be regarded as deformations of the ordinary differential calculus on $\mathbb{R}^n$ and are therefore of special interest. We then address the classification problem for such calculi with constant structure functions and describe corresponding results. The action of the 'exterior derivative' d determines left- and right-partial derivatives $D_{\pm i} : \mathcal{A} \to \mathcal{A}$ via

$$\mathrm{d}f = (D_i f) \, \mathrm{d}x^i = \mathrm{d}x^i \, D_{-i} f \qquad (\forall \, f \in \mathcal{A}). \tag{1.4}$$

They display the most important properties of a differential calculus. Some general results concerning their structure are obtained (see also subsection 3.3). Examples are provided by the irreducible calculi which arose from our classification of $n = 3$ calculi. Section 5 contains some conclusions.

† *On the r.h.s. of the last equation and in the following we are using the summation convention if not stated otherwise.*

‡ In [7] the case $n = 2$ and $C^{k\ell}{}_m$ linear in x^ℓ has been treated. Such differential calculi are also obtained from calculi on the Heisenberg algebra [8] in the limit $\hbar \to 0$.

2. Algebraic differential calculi on associative algebras

In this section some basic algebraic constructions are recalled which are needed in the following sections.

Let $\mathcal{A}$ be an associative algebra over† $\mathbb{C}$ with unit $\mathbf{1}$. A *differential calculus*‡ $(\Omega(\mathcal{A}), \mathrm{d})$ on $\mathcal{A}$ is a graded associative algebra

$$\Omega(\mathcal{A}) = \bigoplus_{r=0}^{\infty} \Omega^r(\mathcal{A}) \tag{2.1}$$

where $\Omega^r(\mathcal{A})$ are $\mathcal{A}$-bimodules and $\Omega^0(\mathcal{A}) = \mathcal{A}$. It is supplied with a linear operator of degree 1

$$\mathrm{d} : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A}) \tag{2.2}$$

satisfying $\mathrm{d}^2 = 0$, $\mathrm{d}\mathbf{1} = 0$ and

$$\mathrm{d}(\omega\omega') = (\mathrm{d}\omega)\,\omega' + (-1)^r \omega\,\mathrm{d}\omega' \tag{2.3}$$

where $\omega \in \Omega^r(\mathcal{A})$. d is called *exterior derivative*. We also demand that, for $r > 0$, $\Omega^r(\mathcal{A})$ is *generated* by d in the sense that $\mathrm{d}\Omega^{r-1}(\mathcal{A})$ generates $\Omega^r(\mathcal{A})$ as an $\mathcal{A}$-bimodule. We also assume that $\Omega(\mathcal{A})$ is unital with unit $(\mathbf{1}, 0, \ldots)$. The elements of $\Omega^r(\mathcal{A})$ are called *r-forms*.

$(\Omega^1(\mathcal{A}), \mathrm{d})$ (with d restricted to $\mathcal{A}$) is a *first-order differential calculus* on $\mathcal{A}$. d is then a derivation $\mathcal{A} \to \Omega^1(\mathcal{A})$.

2.1. The universal first-order differential calculus

The tensor product $\mathcal{A} \otimes \mathcal{A}$ consists of finite linear combinations (with coefficients in $\mathbb{C}$) of terms $f \otimes h$ where $f, h \in \mathcal{A}$. Via

$$g\,(f \otimes h) := (gf) \otimes h \qquad (f \otimes h)\,g := f \otimes (hg) \tag{2.4}$$

it carries an $\mathcal{A}$-bimodule structure. The multiplication in $\mathcal{A}$ yields a map

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad f \otimes h \mapsto fh \tag{2.5}$$

which is a bimodule homomorphism. Defining

$$\tilde{\Omega}^1(\mathcal{A}) := \ker \mu = \left\{ \sum_a f_a \otimes h_a \,\middle|\, \sum_a f_a h_a = 0 \right\} \tag{2.6}$$

there is a map

$$\tilde{\mathrm{d}} : \mathcal{A} \to \tilde{\Omega}^1(\mathcal{A}) \qquad f \mapsto \mathbf{1} \otimes f - f \otimes \mathbb{1}\,. \tag{2.7}$$

The image of $\mathcal{A}$ under $\tilde{\mathrm{d}}$ generates $\tilde{\Omega}^1(\mathcal{A})$ as an $\mathcal{A}$-bimodule. $(\tilde{\Omega}^1(\mathcal{A}), \tilde{\mathrm{d}})$ is the *universal first-order differential calculus* on $\mathcal{A}$. It has the following universal property.

Theorem 2.1. For each derivation $\mathrm{d} : \mathcal{A} \to M$ into some $\mathcal{A}$-bimodule M there is one and only one $\mathcal{A}$-bimodule homomorphism $\phi : \tilde{\Omega}^1(\mathcal{A}) \to M$ such that $\mathrm{d} = \phi \circ \tilde{\mathrm{d}}$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\tilde{\mathrm{d}}"] \arrow[dr, "\mathrm{d}"'] & & \tilde{\Omega}^1(\mathcal{A}) \arrow[dl, "\phi"] \\ & M & \end{tikzcd}$$

† Most of the following also works over $\mathbb{R}$ (or other fields), but for the classification results in section 4 the choice $\mathbb{C}$ is essential.

‡ In the mathematical literature it is usually called a *differential graded algebra*.

Proof. see [10], chapter III, subsection 10.10, for example. □

As a consequence of this theorem, every first-order differential calculus $(\Omega^1(\mathcal{A}), \mathrm{d})$ on $\mathcal{A}$ (which is generated by d) is isomorphic to a quotient of $\tilde{\Omega}^1(\mathcal{A})$ by some $\mathcal{A}$-sub-bimodule (the kernel of the respective homomorphism ϕ).

2.2. *The universal differential calculus*

Let $(\tilde{\Omega}^1(\mathcal{A}), \tilde{\mathrm{d}})$ be the universal first-order differential calculus. Define

$$\tilde{\Omega}^0(\mathcal{A}) := \mathcal{A} \qquad \tilde{\Omega}^p(\mathcal{A}) := \underbrace{\tilde{\Omega}^1(\mathcal{A}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \tilde{\Omega}^1(\mathcal{A})}_{p \text{ times}} . \tag{2.8}$$

Then

$$\tilde{\Omega}(\mathcal{A}) := \bigoplus_{p=0}^{\infty} \tilde{\Omega}^p(\mathcal{A}) \tag{2.9}$$

with the multiplication $\otimes_{\mathcal{A}}$ becomes a graded associative algebra. The extension of $\tilde{\mathrm{d}}$ to an exterior derivative is given by

$$\tilde{\mathrm{d}}(f_0 \otimes f_1 \otimes \cdots \otimes f_p) := \sum_{q=0}^{p+1} (-1)^q f_0 \otimes \cdots \otimes f_{q-1} \otimes \mathbb{1} \otimes f_q \otimes \cdots \otimes f_p \tag{2.10}$$

and $\mathbb{C}$-linearity. $(\tilde{\Omega}(\mathcal{A}), \tilde{\mathrm{d}})$ is the *universal differential calculus*† on $\mathcal{A}$. It has a universal property generalizing theorem 2.1 (see [11], for example). Any differential calculus on $\mathcal{A}$ (for which $\mathrm{d}\Omega^p(\mathcal{A})$ generates $\Omega^{p+1}(\mathcal{A})$ as an $\mathcal{A}$-bimodule) can be obtained from $(\tilde{\Omega}(\mathcal{A}), \tilde{\mathrm{d}})$ as a quotient with respect to some two-sided differential ideal in $\tilde{\Omega}(\mathcal{A})$ (an ideal which is closed under $\tilde{\mathrm{d}}$).

2.3. *Reducibility and skew tensor products of differential calculi*

Let $(\Omega(\mathcal{A}), \mathrm{d})$ and $(\Omega(\mathcal{A}'), \mathrm{d}')$ be differential calculi on $\mathcal{A}$ and $\mathcal{A}'$, respectively. From these one can build the differential calculus $(\Omega(\mathcal{A}) \hat{\otimes} \Omega(\mathcal{A}'), \hat{\mathrm{d}})$, called the *skew tensor product* (cf [11], appendix A, and [12], chapter II ‡). The underlying set is the tensor product $\Omega(\mathcal{A}) \otimes \Omega(\mathcal{A}') =: \hat{\Omega}$. The grading is given by

$$\hat{\Omega} = \bigoplus_{r=0}^{\infty} \hat{\Omega}^r \qquad \text{with} \qquad \hat{\Omega}^r = \bigoplus_{p=0}^{r} \Omega^p(\mathcal{A}) \otimes \Omega^{r-p}(\mathcal{A}') . \tag{2.11}$$

Multiplication is defined by

$$(\omega \hat{\otimes} \omega')(\rho \hat{\otimes} \rho') := (-1)^{\partial\omega' \cdot \partial\rho} (\omega\rho \hat{\otimes} \omega'\rho') \tag{2.12}$$

and $\mathbb{C}$-linearity. $\partial\omega$ denotes the grade of the form ω. We use $\hat{\otimes}$ to stress the difference compared with the canonical multiplication. The linear operator $\hat{\mathrm{d}}$ on $\Omega(\mathcal{A}) \hat{\otimes} \Omega(\mathcal{A}')$ acts as follows:

$$\hat{\mathrm{d}}(\omega \hat{\otimes} \omega') = (\mathrm{d}\omega) \hat{\otimes} \omega' + (-1)^{\partial\omega} \omega \hat{\otimes} \mathrm{d}'\omega' . \tag{2.13}$$

Given a differential calculus on an algebra $\mathcal{A}$, the question arises as to whether it is *reducible* in the sense that it is a skew tensor product of differential calculi. If not, we should call the differential calculus *irreducible*.

† It is Karoubi's *differential envelope* of $\mathcal{A}$ (see [11]).

‡ Instead of *skew* the term *anticommutative* is used there.

2.4. *Inner extensions of derivations*

A derivation $\mathrm{d} : \mathcal{A} \to M$, where M is an $\mathcal{A}$-bimodule, is called *inner* if there is an element $\rho \in M$ such that

$$\mathrm{d}f = [\rho, f] \qquad \forall\, f \in \mathcal{A}. \tag{2.14}$$

We say that a (first-order) differential calculus is inner if its exterior derivative d is inner.

Given a derivation $\mathrm{d} : \mathcal{A} \to M$ (which may already be inner), the $\mathcal{A}$-bimodule M can always be extended into a larger $\mathcal{A}$-bimodule $\check{M}$ such that d becomes inner. This is done by adding an (independent) element ρ as follows. Let ${}_{\mathcal{A}}\rho$ be the free left-$\mathcal{A}$-module generated by ρ and define $\check{M} := M \oplus {}_{\mathcal{A}}\rho$ which is then also a left-$\mathcal{A}$-module. A right-$\mathcal{A}$-module structure can then be introduced on $\check{M}$ by requiring $M \subset \check{M}$ to be an $\mathcal{A}$-sub-bimodule and setting

$$(h\,\rho)\,f := h\,f\,\rho + h\,\mathrm{d}f \qquad \forall\, f, h \in \mathcal{A}. \tag{2.15}$$

Then

$$(h\,\rho)\,f = h\,(f\,\rho + \mathrm{d}f) = h\,(\rho\,f) \tag{2.16}$$

and

$$\rho\,(hf) = h\,f\,\rho + \mathrm{d}(hf) = h\,f\,\rho + h\,\mathrm{d}f + (\mathrm{d}h)\,f = h\,(\rho\,f) + (\mathrm{d}h)\,f = (h\,\rho + \mathrm{d}h)\,f \\ = (\rho\,h)\,f \tag{2.17}$$

which extends the $\mathcal{A}$-bimodule structure of M to $\check{M}$.

In some cases it is possible to enlarge the algebra $\mathcal{A}$ (by introducing an additional generator) to an algebra $\check{\mathcal{A}}$ and to extend d such that it becomes inner with an element ρ of the $\check{\mathcal{A}}$-bimodule generated by $\mathrm{d}\check{\mathcal{A}}$. See subsection 4.1.

3. Differential calculi on commutative algebras

In this section, $\mathcal{A}$ always denotes an associative and *commutative* algebra. It will be shown that the space of 1-forms of any first-order differential calculus on $\mathcal{A}$ then also carries (in a canonical way) the structure of an associative and commutative algebra.

3.1. *The canonical product in the space of universal 1-forms*

With the multiplication

$$(f \otimes h)\,(f' \otimes h') := ff' \otimes hh' \qquad \forall\; f, f', h, h' \in \mathcal{A} \tag{3.1}$$

$\mathcal{A} \otimes \mathcal{A}$ becomes an associative algebra (over $\mathbb{C}$). For a commutative algebra, the map μ defined in (2.5) is an algebra homomorphism. (3.1) thus induces a 'canonical product'

$$\tilde{\Omega}^1(\mathcal{A}) \times \tilde{\Omega}^1(\mathcal{A}) \to \tilde{\Omega}^1(\mathcal{A}) \qquad (\tilde{\omega}, \tilde{\omega}') \mapsto \tilde{\omega} \bullet \tilde{\omega}' \tag{3.2}$$

in the space $\tilde{\Omega}^1(\mathcal{A})$ of 1-forms of the universal first-order differential calculus on $\mathcal{A}$†. It is associative, commutative and $\mathcal{A}$-bilinear, i.e.

$$(f\tilde{\omega}h) \bullet (f'\tilde{\omega}'h') = ff'\,(\tilde{\omega} \bullet \tilde{\omega}')\,hh' \tag{3.3}$$

(cf equation (2.4)). From the identity

$$\left[\sum_a g_a \otimes h_a\,,\; f\right] = \left(\sum_a g_a \otimes h_a\right)(\mathbb{1} \otimes f - f \otimes \mathbb{1}) \tag{3.4}$$

† Here we introduce $\bullet$ in order to distinguish the product from the one in $\tilde{\Omega}(\mathcal{A})$.

in $\mathcal{A}\otimes\mathcal{A}$ and from (2.7) we deduce the following important property:

$$[\tilde{\omega}, f] = \tilde{\omega} \bullet \tilde{\mathrm{d}} f \qquad \forall\, \tilde{\omega} \in \tilde{\Omega}^1(\mathcal{A}) \quad f \in \mathcal{A}\,. \tag{3.5}$$

A simple calculation leads to

$$\tilde{\mathrm{d}}(fh) = f\,\tilde{\mathrm{d}}h + h\,\tilde{\mathrm{d}}f + \tilde{\mathrm{d}}f \bullet \tilde{\mathrm{d}}h \tag{3.6}$$

which is generalized in the following lemma.

Lemma 3.1.

$$\tilde{\mathrm{d}}(f_1\cdots f_r) = \tilde{\mathrm{d}}f_1 \bullet\cdots\bullet \tilde{\mathrm{d}}f_r + \sum_{k=2}^{r} \frac{1}{(k-1)!\,(r-k+1)!}\, f_{(1}\cdots f_{k-1}\,\tilde{\mathrm{d}}f_k \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r)} \tag{3.7}$$

where the indices on the r.h.s. are totally symmetrized (indicated by brackets).

Proof. First we note that

$$f_{(1}\cdots f_{k-1}\,\tilde{\mathrm{d}}f_k \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r+1)} = \sum_{\text{partitions}} (k-1)!\,(r-k+2)!\, f_{\ell_1}\cdots f_{\ell_{k-1}}\,\tilde{\mathrm{d}}f_{\ell_k} \bullet\cdots\bullet \tilde{\mathrm{d}}f_{\ell_{r+1}}$$

where the sum is over all partitions of $(1,\ldots,r+1)$ into ordered tuples $(\ell_1,\ldots,\ell_{k-1})$, $(\ell_k,\ldots,\ell_{r+1})$. This sum splits into a sum over partitions with $r+1 \in \{\ell_1,\ldots,\ell_{k-1}\}$ and a sum over partitions with $r+1 \in \{\ell_k,\ldots,\ell_{r+1}\}$. In the first sum we can use the commutativity of $\mathcal{A}$ to pull f_{r+1} in front of the summation sign. In the second sum the commutativity of $\bullet$ allows us to write $\tilde{d} f_{r+1}$ to the right of all terms. The two sums can then be expressed as a sum over all partitions of $(1,\ldots,r)$ into ordered tuples $(\ell_1,\ldots,\ell_{k-2})$, $(\ell_{k-1},\ldots,\ell_r)$ and $(\ell_1,\ldots,\ell_{k-1})$, $(\ell_k,\ldots,\ell_r)$, respectively. Using our first formula above, we arrive at the identity

$$\begin{aligned} f_{(1}\cdots f_{k-1}\,\tilde{\mathrm{d}}f_k \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r+1)} &= \frac{(k-1)!}{(k-2)!}\, f_{r+1}\, f_{(1}\cdots f_{k-2}\,\tilde{\mathrm{d}}f_{k-1} \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r)} \\ &\quad + \frac{(r+1-k+1)!}{(r-k+1)!}\, f_{(1}\cdots f_{k-1}\,\tilde{\mathrm{d}}f_k \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r)} \bullet \tilde{\mathrm{d}}f_{r+1} \end{aligned}$$

which can now be used to prove (3.7) by induction. □

A result about 2-forms is expressed next.

Lemma 3.2.

$$\tilde{\mathrm{d}}(\tilde{\mathrm{d}}f_1 \bullet\cdots\bullet \tilde{\mathrm{d}}f_r) = -\sum_{k=1}^{r-1} \frac{1}{k!\,(r-k)!}\,(\tilde{\mathrm{d}}f_{(1} \bullet\cdots\bullet \tilde{\mathrm{d}}f_k)\,(\tilde{\mathrm{d}}f_{k+1} \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r)})\,. \tag{3.8}$$

Proof. This is carried out by induction. In order to show that (3.8) implies the corresponding formula with r replaced by $r+1$, one may start with the r.h.s. of the latter and write

$$\begin{aligned} &(\tilde{\mathrm{d}}f_{(1} \bullet\cdots\bullet \tilde{\mathrm{d}}f_k)\,(\tilde{\mathrm{d}}f_{k+1} \bullet\cdots\bullet \tilde{\mathrm{d}}f_{r+1)}) \\ &\qquad = k!\,(r+1-k)! \sum_{\text{partitions}} (\tilde{\mathrm{d}}f_{\ell_1} \bullet\cdots\bullet \tilde{\mathrm{d}}f_{\ell_k})\,(\tilde{\mathrm{d}}f_{\ell_{k+1}} \bullet\cdots\bullet \tilde{\mathrm{d}}f_{\ell_{r+1}}) \end{aligned}$$

where the sum is taken over all partitions of $(1,\ldots,r+1)$ into ordered tuples $(\ell_1,\ldots,\ell_k)$, $(\ell_{k+1},\ldots,\ell_{r+1})$. This sum splits into a sum over partitions with $r+1 \in \{\ell_1,\ldots,\ell_k\}$ and a sum over partitions with $r+1 \in \{\ell_{k+1},\ldots,\ell_{r+1}\}$. The first of these sums can then be expressed as a sum over all partitions of $(1,\ldots,r)$ into ordered tuples $(\ell_1,\ldots,\ell_{k-1})$, $(\ell_k,\ldots,\ell_r)$. The second sum is treated similarly. The further procedure is then quite obvious. □

3.2. The algebra structure of a first-order differential calculus

Let $(\Omega^1(\mathcal{A}), \mathrm{d})$ be a first-order differential calculus on $\mathcal{A}$. In the following, we show that the canonical product in the space $\tilde{\Omega}^1(\mathcal{A})$ of universal 1-forms induces a corresponding product in $\Omega^1(\mathcal{A})$.

Lemma 3.3. If $\phi : \tilde{\Omega}^1(\mathcal{A}) \to \Omega^1(\mathcal{A})$ is an $\mathcal{A}$-bimodule homomorphism, then $\ker\phi$ is an ideal in $\tilde{\Omega}^1(\mathcal{A})$ as an algebra with product $\bullet$.

Proof. An arbitrary element of $\tilde{\Omega}^1(\mathcal{A})$ can be written as $\sum_a f_a(\mathbb{1} \otimes h_a - h_a \otimes \mathbb{1})$ with $f_a, h_a \in \mathcal{A}$. Let $\tilde{\omega} \in \ker\phi$. Then

$$\phi\left(\tilde{\omega} \bullet \sum_a f_a(\mathbb{1} \otimes h_a - h_a \otimes \mathbb{1})\right) = \sum_a f_a\, \phi([\tilde{\omega}, h_a]) = \sum_a f_a\, [\phi(\tilde{\omega}), h_a] = 0\,.$$

Since $\bullet$ is commutative, this shows that $\ker\phi$ is an ideal. □

As a consequence of this lemma and theorem 2.1, we now obtain the following result.

Theorem 3.1. For every first-order differential calculus $(\Omega^1(\mathcal{A}), \mathrm{d})$ there is a unique $\mathcal{A}$-bilinear associative and commutative product $\bullet$ in $\Omega^1(\mathcal{A})$ such that $[\omega, f] = \omega \bullet \mathrm{d}f$. □

The next lemma gives a characterization of inner differential calculi.

Lemma 3.4. The derivation d of a first-order differential calculus is inner if and only if there is a unit with respect to $\bullet$.

Proof. The statement is an immediate consequence of the relation $\omega \bullet \mathrm{d}f = [\omega, f]$ taking into account that $\bullet$ is $\mathcal{A}$-bilinear. □

If a first-order differential calculus is inner, i.e. $\mathrm{d}f = [\rho, f]$ ($\forall\, f \in \mathcal{A}$) with an element $\rho \in \Omega^1(\mathcal{A})$, then ρ is unique. This follows from lemma 3.4 together with the fact that the unit of an algebra is unique. This in turn implies that, if d is inner, the center of the $\mathcal{A}$-bimodule $\Omega^1(\mathcal{A})$ is trivial, i.e. $\{\zeta \in \Omega^1(\mathcal{A}) \mid [\zeta, f] = 0\ \forall\, f \in \mathcal{A}\} = \{0\}$.

Let $\mathcal{I}$ be the two-sided differential ideal in $\tilde{\Omega}(\mathcal{A})$ generated by $\ker\phi$. Now

$$\Omega(\mathcal{A}) := \tilde{\Omega}(\mathcal{A})/\mathcal{I} \tag{3.9}$$

together with $\mathrm{d} := \pi \circ \tilde{\mathrm{d}}$ is a differential calculus on $\mathcal{A}$. Here, $\pi : \tilde{\Omega}(\mathcal{A}) \to \Omega(\mathcal{A})$ is the canonical projection. The ideal $\mathcal{I}$ has a decomposition

$$\mathcal{I} = \bigoplus_{r=0}^{\infty} \mathcal{I}^r \tag{3.10}$$

where $\mathcal{I}^0 = \{0\}$ and $\mathcal{I}^1 = \ker\phi$, so that

$$\Omega(\mathcal{A}) = \bigoplus_{r=0}^{\infty} \Omega^r(\mathcal{A}) \qquad \text{with} \qquad \Omega^r(\mathcal{A}) = \tilde{\Omega}^r(\mathcal{A})/\mathcal{I}^r\,. \tag{3.11}$$

Example. $\tilde{\Omega}^1(\mathcal{A})^{\bullet 2} := \tilde{\Omega}^1(\mathcal{A}) \bullet \tilde{\Omega}^1(\mathcal{A})$ *is an* $\mathcal{A}$*-sub-bimodule and also a two-sided ideal in* $\tilde{\Omega}^1(\mathcal{A})$. Hence

$$\mathcal{K}^1(\mathcal{A}) := \tilde{\Omega}^1(\mathcal{A})/\tilde{\Omega}^1(\mathcal{A})^{\bullet 2} \tag{3.12}$$

carries an $\mathcal{A}$-bimodule structure too and the induced product is trivial, i.e. the product of *any two elements of* $\mathcal{K}^1(\mathcal{A})$ *is equal to zero. Now theorem 3.1 shows that all elements of* $\mathcal{K}^1(\mathcal{A})$ commute with all elements of $\mathcal{A}$. The extension to forms of higher grade is called *Kähler differential calculus* $(\mathcal{K}(\mathcal{A}), \mathrm{d}_{\mathcal{K}})$ on $\mathcal{A}$ [13].

Let $\mathcal{A}$ be the algebra generated by x with the relation $x^N = \mathbb{1}$ for some $N \in \mathbb{N}$. Acting on the latter with the Kähler derivation leads to

$$x^{N-1}\,\mathrm{d}_{\mathcal{K}}x = 0 \tag{3.13}$$

which implies $\mathrm{d}_{\mathcal{K}}x = 0$ so that the Kähler derivation is trivial. In the presence of constraints one is therefore led to consider *non-commutative* differential calculi, where differentials do not commute with elements of $\mathcal{A}$ in general, in order to have a non-trivial d. In particular, this is so for differential calculi on finite sets [6].

Example. Let $\mathcal{A}$ be the commutative and associative algebra which is freely generated by elements t and x (and a unit $\mathbb{1}$). An example of a differential calculus on $\mathcal{A}$ which is not inner is determined by the commutation relations

$$[\mathrm{d}x, x] = \mathrm{d}t \qquad [\mathrm{d}x, t] = 0 = [\mathrm{d}t, t] \tag{3.14}$$

where we assume that $\mathrm{d}x, \mathrm{d}t$ is a basis of $\Omega^1(\mathcal{A})$ as a left- (and right-) $\mathcal{A}$-module. For the associated product we have

$$\mathrm{d}x \bullet \mathrm{d}x = \mathrm{d}t \qquad \mathrm{d}x \bullet \mathrm{d}t = 0 \qquad \mathrm{d}t \bullet \mathrm{d}t = 0\,. \tag{3.15}$$

This product is then consistent with commutativity of differentials and elements of $\mathcal{A}$. A realization of this algebra is given by a stochastic time variable t and a Wiener process $x = W_t$. The above relations are basic formulae in the Itô calculus of stochastic differentials (where d is not a derivation). Our example is easily generalized to the case of several independent Wiener processes. See also [5].

3.3. The case of a freely and finitely generated algebra

Let $\mathcal{A}$ be freely generated by elements x^i, $i = 1, \ldots, n$, and the unit $\mathbb{1}$†. From lemma 3.1 we can then deduce that the 1-forms

$$\tilde{\tau}^{i_1 \cdots i_r} := \tilde{\mathrm{d}}x^{i_1} \bullet \cdots \bullet \tilde{\mathrm{d}}x^{i_r} \qquad (r = 1, \ldots) \tag{3.16}$$

generate $\tilde{\Omega}^1(\mathcal{A})$ as a left-$\mathcal{A}$-module. $\tilde{\tau}^{i_1 \cdots i_r}$ is totally symmetric in the indices $i_1, \ldots, i_r$, so that we should restrict the latter by $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$.

Lemma 3.5. The set of 1-forms

$$B := \{\tilde{\tau}^{i_1 \cdots i_r} \in \tilde{\Omega}^1(\mathcal{A}) \mid i_1 \leqslant \cdots \leqslant i_r,\ r = 1, 2, \ldots\} \tag{3.17}$$

is a basis of $\tilde{\Omega}^1(\mathcal{A})$ as a left-$\mathcal{A}$-module.

† $\mathcal{A}$ consists of finite linear combinations of monomials in $x^1, \ldots, x^n$ and $\mathbb{1}$ with coefficients in $\mathbb{C}$, i.e. $\mathcal{A} = \mathbb{C}[x^1, \ldots, x^n]$. We will not discuss here a possible extension to infinite sums (e.g., the case of analytic functions on $\mathbb{R}^n$).

Proof. We already know that B generates $\tilde{\Omega}^1(\mathcal{A})$ as a left-$\mathcal{A}$-module. It is therefore sufficient to show that any finite subset of B is linearly independent over $\mathcal{A}$. Let

$$0 = \sum_{r=1}^{n} \sum_{i_1 \leqslant i_2 \cdots \leqslant i_r} f_{i_1 \cdots i_r} \tilde{\tau}^{i_1 \cdots i_r}$$

with $f_{i_1 \cdots i_r} \in \mathcal{A}$. Using

$$\tilde{\tau}^{i_1 \cdots i_r} = \mathbf{1} \otimes x^{i_1} \cdots x^{i_r} + \sum_{p=1}^{r-1} \frac{(-1)^p}{p!(r-p)!} x^{(i_1} \cdots x^{i_p} \otimes x^{i_{p+1}} \cdots x^{i_r)} + (-1)^r x^{i_1} \cdots x^{i_r} \otimes \mathbf{1}$$

the above equation leads to

$$0 = \sum_{i_1 \leqslant i_2 \cdots \leqslant i_n} f_{i_1 \cdots i_n} \otimes x^{i_1} \cdots x^{i_n} + \text{rest}$$

where 'rest' consists of a finite sum of tensor products of which the second factor is a monomial of degree $< n$ in the generators x^i of $\mathcal{A}$. Since $\mathcal{A}$ is freely generated, we conclude that $f_{i_1 \cdots i_n} = 0$. By repetition of this argument, $f_{i_1 \cdots i_r} = 0 \ \forall\, i_1, \ldots, i_r,\ r = 1, \ldots, n$. $\square$

Similarly, one can argue that the 2-forms $\tilde{\tau}^{i_1 \cdots i_r} \tilde{\tau}^{j_1 \cdots j_s}$ $(r, s = 1, \ldots)$ constitute a basis of $\tilde{\Omega}^2(\mathcal{A})$ as a left-$\mathcal{A}$-module (see also lemma 3.2), and correspondingly for $\tilde{\Omega}^r(\mathcal{A})$ with $r > 2$.

As a consequence of the preceding lemma

$$\tilde{\mathrm{d}} f = \sum_{r=1}^{\infty} (\tilde{D}_{i_1 \cdots i_r} f)\, \tilde{\tau}^{i_1 \cdots i_r} . \tag{3.18}$$

with operators $\tilde{D}_{i_1 \cdots i_r} : \mathcal{A} \to \mathcal{A}$, where the indices are totally symmetric. Inserted in (3.6) this leads to

$$\tilde{D}_{i_1 \cdots i_r}(fh) = f\, \tilde{D}_{i_1 \cdots i_r} h + h\, \tilde{D}_{i_1 \cdots i_r} f + \sum_{k=1}^{r-1} \frac{1}{k!(r-k)!} (\tilde{D}_{(i_1 \cdots i_k} f)(\tilde{D}_{i_{k+1} \cdots i_r)} h) \tag{3.19}$$

which, in particular, shows that the operators $\tilde{D}_i$ are derivations. As a consequence of (3.18) they satisfy $\tilde{D}_j x^i = \delta^i_j$ and therefore coincide with the ordinary partial derivatives†

$$\tilde{D}_i = \partial_i . \tag{3.20}$$

Applying $\tilde{\mathrm{d}}$ to (3.18) using (3.8) in the form

$$\tilde{\mathrm{d}} \tilde{\tau}^{i_1 \cdots i_r} = -\sum_{k=1}^{r-1} \frac{1}{k!(r-k)!} \tilde{\tau}^{(i_1 \cdots i_k} \tilde{\tau}^{i_{k+1} \cdots i_r)} \tag{3.21}$$

and $\tilde{\mathrm{d}}^2 = 0$, we obtain

$$\tilde{D}_{i_1 \cdots i_k} \tilde{D}_{i_{k+1} \cdots i_r} = \binom{r}{k} \tilde{D}_{i_1 \cdots i_r} . \tag{3.22}$$

Together with (3.20) this implies

$$\tilde{D}_{i_1 \cdots i_r} = \frac{1}{r!} \partial_{i_1} \cdots \partial_{i_r} \qquad (r = 1, \ldots) . \tag{3.23}$$

So far we have treated the case of the universal differential calculus with its algebra structure. For any other first-order differential calculus $(\Omega^1(\mathcal{A}), \mathrm{d})$ we can define $\tau^{i_1 \cdots i_r}$ as in (3.16) (with $\tilde{\mathrm{d}} x^i$ replaced by $\mathrm{d}x^i$). These 1-forms are then, however, not linearly independent. Nevertheless, the formulae derived above induce corresponding formulae for any differential calculus, as demonstrated in the following examples.

† The 'ordinary partial derivatives' are the derivations $\partial_k : \mathcal{A} \to \mathcal{A}$ $(k = 1, \ldots, n)$ with $\partial_k x^\ell = \delta^\ell_k$.

Examples. We evaluate (3.18) with (3.23) for some examples of differential calculi.

(1) For the Kähler calculus, where $\mathrm{d}x^i \bullet \mathrm{d}x^j = 0$, we recover the familiar formula $\mathrm{d}f = (\partial_i f)\,\mathrm{d}x^i$.

(2) The lattice calculus in [3] is determined by $\mathrm{d}x^i \bullet \mathrm{d}x^j = \ell\,\delta^{ij}\,\mathrm{d}x^j$ (no summation, $\ell \in \mathbb{R} \setminus \{0\}$). In this case we obtain

$$\begin{aligned}\mathrm{d}f &= \sum_{i=1}^{n} \frac{1}{\ell}\,(\exp(\ell\partial_i) - 1) f\,\mathrm{d}x^i \\ &= \sum_{i=1}^{n} \frac{1}{\ell}\,[f(x^1, \ldots, x^i + \ell, \ldots, x^n) - f(x)]\,\mathrm{d}x^i\,.\end{aligned}$$

In the universal differential calculus, the ideal by which we have to factorize $\tilde{\Omega}^1(\mathcal{A})$ in order to obtain the calculus under consideration is generated by $\tilde{\mathrm{d}}x^i \bullet \tilde{\mathrm{d}}x^j - \ell\,\delta^{ij}\,\tilde{\mathrm{d}}x^j$. Representing the x^i as coordinate functions on $\mathbb{C}^n$ (or $\mathbb{R}^n$) and evaluating the last expressions on $(a, b) \in \mathbb{C}^n \times \mathbb{C}^n$ using (2.7), we find

$$(\tilde{\mathrm{d}}x^i \bullet \tilde{\mathrm{d}}x^j - \ell\,\delta^{ij}\,\tilde{\mathrm{d}}x^j)(a, b) = (b^j - a^j)[b^i - a^i - \ell\delta^{ij}]\,.$$

Equated to zero, this precisely displays the lattice structure.

(3) For $n = 1$ (for simplicity), the symmetric lattice calculus discussed in [6] can be defined by $\mathrm{d}x \bullet \mathrm{d}x \bullet \mathrm{d}x = \ell^2\,\mathrm{d}x$. Then

$$\begin{aligned}\mathrm{d}f &= \sum_{r=0}^{\infty} \frac{\ell^{2r}}{(2r+1)!}\,(\partial^{2r+1} f)\,\mathrm{d}x + \sum_{r=1}^{\infty} \frac{\ell^{2(r-1)}}{(2r)!}\,(\partial^{2r} f)\,\mathrm{d}x \bullet \mathrm{d}x \\ &= \bar{\partial} f\,\mathrm{d}x + \tfrac{1}{2}\,\Delta f\,\mathrm{d}x \bullet \mathrm{d}x\end{aligned}$$

where

$$\bar{\partial} f := \frac{1}{2\ell}\,[f(x+\ell) - f(x-\ell)] \qquad \Delta f := \frac{1}{\ell^2}\,[f(x+\ell) + f(x-\ell) - 2\,f(x)]\,.$$

With x as a coordinate function on $\mathbb{C}$, we find

$$(\tilde{\mathrm{d}}x \bullet \tilde{\mathrm{d}}x \bullet \tilde{\mathrm{d}}x - \ell^2\,\tilde{\mathrm{d}}x)(a, b) = (b-a)[(b-a)^2 - \ell^2]\,.$$

Equated to zero, this implies $b = a$ or $b = a + \ell$ or $b = a - \ell$ which reveals the symmetric lattice structure (see [6]).

For $\ell = 0$ one obtains

$$\mathrm{d}f = \partial f\,\mathrm{d}x + \tfrac{1}{2}\,\partial^2 f\,\mathrm{d}x \bullet \mathrm{d}x\,.$$

The last type of calculus appears in the classical limit ($q \to 1$) of a bicovariant differential calculus on the quantum groups $SL_q(n)$ [14, 5, 15]. Via $\mathrm{d}x \bullet \mathrm{d}x \mapsto \mathrm{d}t$ contact is made with the calculus of the last example in the previous subsection.

(4) Generalizing the last two examples for $n = 1$, we consider the ideal in $\tilde{\Omega}^1(\mathcal{A})$ generated by

$$(\tilde{\mathrm{d}}x)^{\bullet(k+1)} - \ell^k\,\tilde{\mathrm{d}}x$$

for some fixed $k \in \mathbb{N}$. Evaluated on $(a, b) \in \mathbb{C} \times \mathbb{C}$, it leads us to the equation

$$(b-a)[(b-a)^k - \ell^k] = 0\,.$$

This defines an algorithm which, fixing a starting point a, generates new points $a + \ell q^r$ for $r = 0, 1, \ldots, k$, where q is a primitive kth root of unity. In this way, a lattice is created in the complex plane and the differential calculus can be restricted to (the functions on) it.

Using $\sum_{j=0}^{k-1} q^j = (q^k - 1)/(q - 1) = 0$ for $k > 1$, we find

$$
\begin{aligned}
\mathrm{d}f &= \sum_{r=0}^{\infty} \frac{\ell^{kr}}{(kr+1)!} (\partial^{kr+1} f)\, \mathrm{d}x + \sum_{r=0}^{\infty} \frac{\ell^{kr}}{(kr+2)!} (\partial^{kr+2} f)\, \mathrm{d}x \bullet \mathrm{d}x \\
&\quad + \cdots + \sum_{r=0}^{\infty} \frac{\ell^{kr}}{(kr+k)!} (\partial^{kr+k} f)\, (\mathrm{d}x)^{\bullet k} \\
&= \sum_{j=1}^{k-1} (D_j f)\, (\mathrm{d}x)^{\bullet j} + (D_k - \ell^{-k}) f\ (\mathrm{d}x)^{\bullet k}
\end{aligned}
$$

with

$$
D_j f = \frac{1}{k\ell^j} \sum_{m=0}^{k-1} q^{j(k-m)}\, f(x + \ell q^m) .
$$

In terms of the 1-forms

$$
\theta^j := \frac{1}{k} \sum_{m=1}^{k} q^{j(k-m)}\, \ell^{-m}\, (\mathrm{d}x)^{\bullet m}
$$

this becomes

$$
\mathrm{d}f = \sum_{j=0}^{k-1} [f(x + \ell q^j) - f(x)]\, \theta^j .
$$

Furthermore, we have the following simple commutation relations:

$$
\theta^j f(x) = f(x + \ell q^j) \theta^j .
$$

For $k = 3$ the lattice is triangular†. More precisely, it should be regarded as an oriented lattice. For $k = 4$ one obtains the corresponding symmetric lattice.

4. Deformations of the ordinary differential calculus on freely generated commutative algebras

Throughout this section $\mathcal{A}$ denotes an associative and commutative algebra which is freely generated by $x^1, \ldots, x^n$ and the unit $\mathbb{1}$. Furthermore, we restrict our considerations to *n-dimensional* first-order differential calculi $(\Omega^1(\mathcal{A}), \mathrm{d})$. For these calculi the differentials $\mathrm{d}x^i$, $i = 1, \ldots, n$, form a basis of $\Omega^1(\mathcal{A})$ as a left-$\mathcal{A}$-module. Such calculi may be regarded as (algebraic) deformations of the ordinary (Kähler) differential calculus and are therefore of particular interest. As a consequence of these assumptions

$$
\mathrm{d}x^i \bullet \mathrm{d}x^j = [\mathrm{d}x^i, x^j] = C^{ij}{}_k\, \mathrm{d}x^k \tag{4.1}
$$

with $C^{ij}{}_k \in \mathcal{A}$. The commutativity and associativity of $\bullet$ imply

$$
C^{ij}{}_k\, \mathrm{d}x^k = \mathrm{d}x^i \bullet \mathrm{d}x^j = \mathrm{d}x^j \bullet \mathrm{d}x^i = C^{ji}{}_k\, \mathrm{d}x^k \tag{4.2}
$$

† It is of the kind which underlies the hard hexagon model in statistical mechanics [16].

and

$$C^{ij}{}_\ell\, C^{\ell k}{}_m\, \mathrm{d}x^m = C^{ij}{}_\ell\, \mathrm{d}x^\ell \bullet \mathrm{d}x^k = (\mathrm{d}x^i \bullet \mathrm{d}x^j) \bullet \mathrm{d}x^k = \mathrm{d}x^i \bullet (\mathrm{d}x^j \bullet \mathrm{d}x^k)$$
$$= \mathrm{d}x^i \bullet C^{jk}{}_\ell\, \mathrm{d}x^\ell = C^{jk}{}_\ell\, C^{i\ell}{}_m\, \mathrm{d}x^m \tag{4.3}$$

which lead to the *consistency conditions*

$$C^{ij}{}_k = C^{ji}{}_k \qquad C^{ik}{}_\ell\, C^{j\ell}{}_m = C^{jk}{}_\ell\, C^{i\ell}{}_m \tag{4.4}$$

(see also [3]). In terms of the (structure) matrices $\mathbf{C}^k$ with entries $(\mathbf{C}^k)^i{}_j := C^{ki}{}_j$ the first of these conditions means that the jth row of $\mathbf{C}^i$ equals the ith row of $\mathbf{C}^j$. The second condition says that the $\mathbf{C}^i$ commute with each other:

$$\mathbf{C}^i\, \mathbf{C}^j = \mathbf{C}^j\, \mathbf{C}^i\,. \tag{4.5}$$

Furthermore, $\mathbf{C}^i \mathbf{C}^j = C^{ij}{}_k\, \mathbf{C}^k$. The matrices $\mathbf{C}^i$ thus provide us with a representation of the algebra $(\Omega^1(\mathcal{A}), \bullet)$. As a consequence of the foregoing, the classification of first-order differential calculi of the kind specified above with $C^{ij}{}_k \in \mathbb{C}$ is equivalent to the classification of commutative and associative algebras over $\mathbb{C}$.

Remark. More generally, when the conditions (4.4) are satisfied, (4.1) determines a (first-order) differential calculus on any algebra $\mathcal{A}$ which is freely generated by the x^i modulo commutation relations such that $[x^i, x^j]$ is constant with respect to d (for all i, j). Special examples are the Heisenberg algebras of quantum mechanics (see also [1,8] for related work). Further examples are the algebras considered in [17] where $[x^k, x^\ell] = \mathrm{i}\, Q^{k\ell}$ with an antisymmetric tensor operator Q^{ij} which is central in the algebra generated by the x^k†. The solutions of the consistency conditions presented in subsections 4.3 and 4.4 therefore also determine differential calculi on such non-commutative algebras.

In the following subsections we first introduce a notion of 'extension' of a differential calculus (following the general receipe of subsection 2.4). A procedure for the classification of differential calculi with constant structure functions is then outlined and applied to the cases where $n = 2$ and $n = 3$. The action of an exterior derivative on $\mathcal{A}$ is determined by left- (or right-) partial derivatives, for which we derive some general formulae and which we calculate for several examples of differential calculi. Particular solutions of the consistency conditions for arbitrary n are discussed in the last two subsections.

4.1. Inner differential calculi and inner extensions of differential calculi

The following result gives a criterion for a differential calculus to be inner (in the sense of subsection 2.4).

Lemma 4.1. d is inner if and only if there is a 1-form $\rho = \rho_k\, \mathrm{d}x^k$ with $\rho_k\, \mathbf{C}^k = \mathbf{1}$ (the unit $n \times n$ matrix).

Proof.
'$\Rightarrow$': follows immediately from (4.1) and $\mathrm{d}x^i = [\rho, x^i]$.
'$\Leftarrow$':

$$\mathrm{d}f = \delta^i_j\, (D_i f)\, \mathrm{d}x^j = \rho_k\, (D_i f)\, C^{ik}{}_j\, \mathrm{d}x^j = \rho_k\, (D_i f)\, [\mathrm{d}x^i, x^k] = \rho_k\, [\mathrm{d}f, x^k]$$
$$= \rho_k\, [\mathrm{d}x^k, f] = [\rho, f]\,.$$

□

† Here we have to make the assumption that $Q^{k\ell}$ is annihilated by d.

Let $(\Omega^1(\mathcal{A}), \mathrm{d})$ be a first-order differential calculus. To the generators $x^1, \ldots, x^n$ of $\mathcal{A}$ we adjoin an element x^{n+1} to freely generate the larger commutative algebra $\check{\mathcal{A}} = \mathcal{A}[x^{n+1}]$. On the latter we introduce an $(n+1)$-dimensional first-order differential calculus via structure matrices as follows. Define

$$\check{\mathbf{C}}^i := \left(\begin{array}{c|c} & 0 \\ \mathbf{C}^i & \vdots \\ & 0 \\ \hline e^i & 0 \end{array} \right) \qquad (i = 1, \ldots, n) \tag{4.6}$$

where $\mathbf{C}^i$ are the structure matrices of $(\Omega^1(\mathcal{A}), \mathrm{d})$ and e^i is the row vector with entries $e^i_j = \delta^i_j$, $j = 1, \ldots, n$. Let $\check{\mathbf{C}}^{n+1}$ be the $(n+1) \times (n+1)$ unit matrix. The matrices $\check{\mathbf{C}}^I$, $I = 1, \ldots, n+1$, then satisfy the consistency conditions (4.4) (if the $\mathbf{C}^i$ satisfy them). For the enlarged differential calculus $(\Omega^1(\check{\mathcal{A}}), \check{\mathrm{d}}) =: \mathbf{Ext}(\Omega^1(\mathcal{A}), \mathrm{d})$ the extended derivation is inner

$$\check{\mathrm{d}} f = [\check{\mathrm{d}} x^{n+1}, f] \qquad (\forall\, f \in \check{\mathcal{A}}) \tag{4.7}$$

i.e. we have (2.14) with $\rho = \check{\mathrm{d}} x^{n+1}$. In particular, if $(\Omega^1(\mathcal{A}), \mathrm{d})$ is not inner, then there is always an extension of it which is inner. This observation is helpful since it is often much easier to carry out calculations with an inner exterior derivative.

4.2. *Procedure for classification of constant structure functions*

With the additional assumption that the structure functions are constant, i.e. $C^{ij}{}_k \in \mathbb{C}$, it is in principle possible to classify all first-order differential calculi†. This has been done in [3] for the case $n = 2$. However, the methods used there are not applicable to the case $n > 2$, in contrast to the procedure which we outline below, which is then applied to the cases $n = 2$ and $n = 3$.

Under a $GL(n, \mathbb{C})$-transformation

$$x'^k = U^k{}_\ell\, x^\ell \qquad \text{with} \qquad U = (U^k{}_\ell) \in GL(n, \mathbb{C}) \tag{4.8}$$

the commutation relations (4.1) are invariant if

$$C'^{ij}{}_k = U^i{}_r\, U^j{}_s\, C^{rs}{}_t\, (U^{-1})^t{}_k \tag{4.9}$$

and

$$\mathbf{C}'^i = U^i{}_j\, (U\, \mathbf{C}^j\, U^{-1}) \tag{4.10}$$

respectively. These transformations preserve conditions (4.4). In order to classify differential calculi one should therefore determine all equivalence classes of structure matrices with respect to $GL(n, \mathbb{C})$-transformations. Thanks to the commutativity of the matrices $\mathbf{C}^i$, there is a $U \in GL(n, \mathbb{C})$ such that, for all $i = 1, \ldots, n$, the $U\mathbf{C}^i U^{-1}$ are triangular, i.e. have zeros everywhere above the diagonal. This is a consequence of the Jordan trigonalization theorem. But then also the $\mathbf{C}'^i$ in (4.10) are triangular as linear combinations of triangular matrices.

† All first-order differential calculi with constant structure functions extend to higher orders with the usual anticommutation rule for the product of differentials, $\mathrm{d}x^i\, \mathrm{d}x^j = -\mathrm{d}x^j\, \mathrm{d}x^i$. Of course, this simple rule does not extend to arbitrary 1-forms in case of a non-commutative differential calculus (where some of the $C^{ij}{}_k$ are different from zero).

Hence, in each $GL(n,\mathbb{C})$ orbit of structure matrices there are representatives which are triangular and only for those we have to solve the $GL(n,\mathbb{C})$-invariant conditions (4.4). The symmetry condition now reduces the $\mathbf{C}^i$ to the following form

$$\mathbf{C}^1 = \begin{pmatrix} C^{11}{}_1 & 0 & \cdots & 0 \\ C^{12}{}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ C^{1n}{}_1 & 0 & \cdots & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} C^{21}{}_1 & 0 & 0 & \cdots & 0 \\ C^{22}{}_1 & C^{22}{}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C^{2n}{}_1 & C^{2n}{}_2 & 0 & \cdots & 0 \end{pmatrix}$$

$$\cdots \qquad \mathbf{C}^n = \begin{pmatrix} C^{n1}{}_1 & 0 & \cdots & 0 \\ C^{n2}{}_1 & C^{n2}{}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ C^{nn}{}_1 & C^{nn}{}_2 & \cdots & C^{nn}{}_n \end{pmatrix}. \tag{4.11}$$

Although we made use of the fact that the $\mathbf{C}^i$ have to commute with each other in order to derive (4.11), the commutativity is not yet built in completely. The further procedure may now be as follows. There are $GL(n,\mathbb{C})$-transformations which preserve the above form of the matrices $\mathbf{C}^i$. They can be used to further simplify their structure. The remaining complex constants are then constrained by quadratic equations resulting from the condition (4.5) that the $\mathbf{C}^i$ have to commute with each other. These equations have to be solved.

In the simple case $n=1$, where $[\mathrm{d}x, x] = c\,\mathrm{d}x$, there are two orbits. $c=0$ represents the ordinary (Kähler) differential calculus. We refer to it as $\boxed{\mathbf{K}}$. The other orbit where $c \neq 0$ can be represented by $c=1$. It describes a differential calculus on a one-dimensional lattice† denoted by $\boxed{\mathbf{L}}$.

From these one-dimensional calculi one can build differential calculi on algebras with more than one generator. The general construction has been recalled in subsection 2.3. Let $y^1,\ldots,y^r$ and $z^1,\ldots,z^s$ be the generators of two commutative algebras with, respectively, r- and s-dimensional (first-order) differential calculi determined by

$$[\mathrm{d}y^a, y^b] = C^{ab}{}_c\,\mathrm{d}y^c \qquad [\mathrm{d}z^{a'}, z^{b'}] = C^{a'b'}{}_{c'}\,\mathrm{d}z^{c'}\,. \tag{4.12}$$

For $x^a := y^a \otimes \mathbb{1}$ and $x^{r+a'} := \mathbb{1} \otimes z^{a'}$ this implies

$$[\mathrm{d}x^i, x^j] = \hat{C}^{ij}{}_k\,\mathrm{d}x^k \qquad (i,j = 1,\ldots,r+s) \tag{4.13}$$

where $\hat{C}^{ab}{}_c = C^{ab}{}_c$, $\hat{C}^{r+a',r+b'}{}_{r+c'} = C^{a'b'}{}_{c'}$ and $\hat{C}^{ij}{}_k = 0$ otherwise. Conversely, if after some $GL(n,\mathbb{C})$-transformation the structure matrices of a differential calculus decompose in this way, the calculus is reducible and can be expressed as a skew tensor product of lower-dimensional calculi.

The n-dimensional irreducible calculi can be further classified into those which are extensions (in the sense of subsection 4.1) of $(n-1)$-dimensional calculi and those which are not. This makes sense on the basis of the following result.

Lemma 4.2. The extensions of all representatives of a $GL(n,\mathbb{C})$ orbit of n-dimensional differential calculi lie in the same $GL(n+1,\mathbb{C})$ orbit.

† To see this, one actually has to go beyond the algebra of polynomials since functions with period c play an essential role in this case [3].

Proof. With the special $GL(n+1,\mathbb{C})$ matrix

$$\check{U} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

where $U \in GL(n,\mathbb{C})$ we find

$$\check{\mathbf{C}}'^i = \check{U}^i{}_J\, \check{U}\check{\mathbf{C}}^J \check{U}^{-1} = U^i{}_j \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{C}^j & 0 \\ e^j & 0 \end{pmatrix} \begin{pmatrix} U^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} U^i{}_j\, U\mathbf{C}^j U^{-1} & 0 \\ U^i{}_j\, e^j\, U^{-1} & 0 \end{pmatrix}.$$

Since $U^i{}_j\, e^j\, U^{-1} = e^i$, this is the extension of the U-transformed $\mathbf{C}^i$. Furthermore, $\check{\mathbf{C}}'^{n+1} = \check{\mathbf{C}}^{n+1}$ (which is the $(n+1)\times(n+1)$ unit matrix). □

4.3. Classification of two-dimensional differential calculi

For $n = 2$ equation (4.11) becomes

$$\mathbf{C}^1 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} b & 0 \\ c & d \end{pmatrix}. \tag{4.14}$$

The two matrices commute iff the complex constants a, b, c, d are related by

$$b^2 - a\,c - b\,d = 0\,. \tag{4.15}$$

An arbitrary element of $GL(2,\mathbb{C})$ is given by

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} \tag{4.16}$$

with $\mathcal{D} := sv - tu \neq 0$. It acts on the matrices $\mathbf{C}^i$ as follows

$$\mathbf{C}'^1 = \frac{1}{\mathcal{D}} \begin{pmatrix} s^2va + 2stvb + t^2vc - t^2ud & -s^2ta - 2st^2b - t^3c + st^2d \\ suva + v(sv+tu)b + tv^2c - tuvd & -stua - t(sv+tu)b - t^2vc + stvd \end{pmatrix}$$

$$\mathbf{C}'^2 = \frac{1}{\mathcal{D}} \begin{pmatrix} suva + v(tu+sv)b + tv^2c - tuvd & -stua - t(tu+sv)b - t^2vc + stvd \\ u^2va + 2uv^2b + v^3c - uv^2d & -tu^2a - 2tuvb - tv^2c + sv^2d \end{pmatrix}. \tag{4.17}$$

For $t = 0$ this transformation preserves the form of the matrices in (4.14)

$$\mathbf{C}'^1 = \begin{pmatrix} sa & 0 \\ ua + vb & 0 \end{pmatrix} \qquad \mathbf{C}'^2 = \begin{pmatrix} ua + vb & 0 \\ \frac{1}{s}(u^2a + 2uvb + v^2c - uvd) & vd \end{pmatrix}. \tag{4.18}$$

It can thus be used to further reduce the parameter freedom of the matrices $\mathbf{C}^i$.

If $a \neq 0$, we set $s = 1/a$ and $u = -vb/a$. Then

$$\mathbf{C}'^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}'^2 = \begin{pmatrix} 0 & 0 \\ 0 & vd \end{pmatrix} \tag{4.19}$$

using (4.15). If $d = 0$ we have $\mathbf{C}'^2 = 0$. Otherwise the choice $v = 1/d$ leads to

$$\mathbf{C}'^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.20}$$

If $a = 0$ and $b = 0$, so that $\mathbf{C}'^1 = 0$, we can arrange either $\mathbf{C}'^2 = 0$, $\mathbf{C}'^2$ of the form (4.20), or

$$\mathbf{C}'^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.21}$$

In the remaining case $a = 0$ and $b \neq 0$ one can always reach

$$\mathbf{C}'^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{C}'^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.22}$$

In all these cases, (4.15) is automatically satisfied. It has still to be checked, with the help of (4.17), which of the representatives for $\mathbf{C}^1$ and $\mathbf{C}^2$ obtained in this way generate different orbits. In the following we list representatives from all distinct orbits. The respective complete orbit is then obtained via (4.17)†.

(1) For $\mathbf{C}^1 = \mathbf{C}^2 = 0$ we recover the commutative (Kähler) differential calculus. It is reducible since it is the skew tensor product of two one-dimensional commutative differential calculi: $\boxed{\mathbf{K}} \hat{\otimes} \boxed{\mathbf{K}}$.

(2) The pair of matrices

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

represents $\boxed{\mathbf{K}} \hat{\otimes} \boxed{\mathbf{L}}$.

(3) The matrix pair

$$\mathbf{C}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

corresponds to $\boxed{\mathbf{L}} \hat{\otimes} \boxed{\mathbf{L}} = \mathbf{Ext}(\boxed{\mathbf{L}})$.

(4) A further calculus is given by

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is neither reducible nor the extension of a one-dimensional calculus. It therefore plays a role as a 'building block' for the construction of higher-dimensional differential calculi. We will refer to it as $\boxed{\ \mathbf{I}\ }$. This calculus is a special case of a class of calculi which has been investigated in [4, 5].

(5) Another irreducible calculus is determined by

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is the extension of $\boxed{\mathbf{K}}$ and shall hence be denoted as $\mathbf{Ext}(\boxed{\mathbf{K}})$.

† The pair of matrices $\mathbf{C}^i$ with $a = 1$ and $b = c = d = 0$ which we encountered above lies in the orbit of solution (2).

If we want to have an involution on the differential algebra, we have to decompose the calculi into orbits with respect to the action of $GL(2,\mathbb{R})$. The $GL(2,\mathbb{C})$ orbit of (3) then splits into two $GL(2,\mathbb{R})$ orbits (cf [3]).

4.4. Classification of three-dimensional differential calculi

In this subsection we apply the procedure described in subsection 4.2 to the case of an algebra with three generators. (4.11) then reads

$$\mathbf{C}^1 = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} b & 0 & 0 \\ d & e & 0 \\ f & g & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} c & 0 & 0 \\ f & g & 0 \\ h & k & l \end{pmatrix}. \tag{4.23}$$

The complex entries are subject to the relations

$$\begin{aligned} &b^2 - be - ad = 0 \\ &bc - af - bg = 0 \\ &c^2 - ah - bk - cl = 0 \\ &cd + ef - bf - dg = 0 \\ &cf + fg - bh - dk - fl = 0 \\ &g^2 - ek - gl = 0\,. \end{aligned} \tag{4.24}$$

Proceeding as in the two-dimensional case treated in the previous subsection, after a tedious calculation one ends up with the following list of representatives of $GL(3,\mathbb{C})$ orbits†.

(1) $\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{K}}$.

(2) $\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{L}}$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(3) $\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{L}}\,\hat{\otimes}\,\boxed{\mathbf{L}}$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(4) $\boxed{\mathbf{L}}\,\hat{\otimes}\,\boxed{\mathbf{L}}\,\hat{\otimes}\,\boxed{\mathbf{L}} = \mathbf{Ext}(\boxed{\mathbf{L}}\,\hat{\otimes}\,\boxed{\mathbf{L}})$

$$\mathbf{C}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(5) $\boxed{\;\mathbf{I}\;}\,\hat{\otimes}\,\boxed{\mathbf{K}}$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

† Some more details are presented in [9].

(6) $\boxed{\;\mathbf{I}\;}\,\hat{\otimes}\,\boxed{\mathbf{L}}$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(7) $\mathbf{Ext}(\boxed{\mathbf{K}})\,\hat{\otimes}\,\boxed{\mathbf{K}}$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(8) $\mathbf{Ext}(\boxed{\mathbf{K}})\,\hat{\otimes}\,\boxed{\mathbf{L}} = \mathbf{Ext}(\mathbf{Ext}(\boxed{\mathbf{K}})) = \mathbf{Ext}(\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{L}})$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(9) An irreducible calculus is given by

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(10) Another irreducible calculus is determined by

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(11) $\mathbf{Ext}(\boxed{\mathbf{K}}\,\hat{\otimes}\,\boxed{\mathbf{K}})$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(12) $\mathbf{Ext}(\boxed{\;\mathbf{I}\;})$

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The last four of these calculi are irreducible. Only two calculi, (9) and (10), are new in the sense that they cannot be obtained as a skew tensor product or an extension of lower-dimensional calculi. We shall see in subsection 4.5 that (9) is a special case of the calculus explored in [4, 5] for arbitrary n, to which the two-dimensional calculus $\boxed{\;\mathbf{I}\;}$ also belongs. A generalization of (10) to arbitrary n is presented in subsection 4.6.

4.5. Left- and right-partial derivatives

Left-partial derivatives are defined as $\mathbb{C}$-linear maps $D_j : \mathcal{A} \to \mathcal{A}$ by

$$\mathrm{d}f =: (D_j f)\,\mathrm{d}x^j \qquad \forall\, f \in \mathcal{A}. \tag{4.25}$$

Using (4.1), one finds

$$\begin{aligned}[\mathrm{d}f, h] &= (D_i f)\,[\mathrm{d}x^i, h] = (D_i f)\,[\mathrm{d}h, x^i] = (D_i f)(D_j h)\,[\mathrm{d}x^j, x^i] \\ &= (D_i f)(D_j h)\, C^{ij}{}_k\,\mathrm{d}x^k .\end{aligned} \tag{4.26}$$

This leads to

$$\begin{aligned} D_j(fh)\,dx^j &= \mathrm{d}(fh) = f\,\mathrm{d}h + h\,\mathrm{d}f + [\mathrm{d}f, h] \\ &= \{(D_j f)\,h + f\,D_j h + (D_k f)(D_\ell h)\,C^{k\ell}{}_j\}\,\mathrm{d}x^j \qquad \forall\, f, h \in \mathcal{A} \end{aligned} \tag{4.27}$$

from which we can read off a twisted Leibniz rule for the D_j.

Lemma 4.3. The left-partial derivatives are given by

$$D_j = \sum_{r=1}^{\infty} \frac{1}{r!}\,(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-1}})^{k_r}{}_j\,\partial_{k_1}\cdots\partial_{k_r} \tag{4.28}$$

in terms of ordinary partial derivatives†.

Proof. First we note that $(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-1}})^{k_r}{}_j$ is totally symmetric in the indices $k_1, \ldots, k_r$ as a consequence of $C^{ij}{}_k = C^{ji}{}_k$ and the commutativity of the structure matrices $\mathbf{C}^i$. Because of the $\mathbb{C}$-linearity of the D_j it is sufficient to prove (4.28) on monomials in x^i, $i = 1, \ldots, n$. This will be done using induction with respect to the degree of monomials. Applied to x^i the formula is obviously true. Let us assume that it holds for monomials up to degree m. If u is a monomial of degree m, then

$$\begin{aligned} &\sum_{r=1}^{\infty} \frac{1}{r!}\,(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-1}})^{k_r}{}_j\,\partial_{k_1}\cdots\partial_{k_r}(x^i u) \\ &\quad = \partial_j(x^i u) + \sum_{r=2}^{\infty} \frac{1}{(r-1)!}\,(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-1}})^{i}{}_j\,\partial_{k_1}\cdots\partial_{k_{r-1}} u \\ &\qquad + x^i \sum_{r=2}^{\infty} \frac{1}{r!}\,(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-1}})^{k_r}{}_j\,\partial_{k_1}\cdots\partial_{k_r} u \\ &\quad = \delta^i_j\,u + \sum_{r=2}^{\infty} \frac{1}{(r-1)!}\,(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-2}})^{k_{r-1}}{}_\ell\,C^{i\ell}{}_j\,\partial_{k_1}\cdots\partial_{k_{r-1}} u + x^i\,D_j u \\ &\quad = \delta^i_j\,u + C^{i\ell}{}_j \sum_{r=1}^{\infty} \frac{1}{r!}\,(\mathbf{C}^{k_1}\cdots\mathbf{C}^{k_{r-1}})^{k_r}{}_\ell\,\partial_{k_1}\cdots\partial_{k_r} u + x^i\,D_j u \\ &\quad = \delta^i_j\,u + C^{i\ell}{}_j\,D_\ell u + x^i\,D_j u \\ &\quad = (D_j x^i)\,u + x^i\,D_j\,u + C^{k\ell}{}_j\,(D_k x^i)\,D_\ell u = D_j(x^i u) \end{aligned}$$

where we used (4.27) and $D_j x^i = \delta^i_j$ in the last steps. □

† The first summand on the r.h.s. is ∂_j.

Remark. For any first-order differential calculus $(\Omega^1(\mathcal{A}), \mathrm{d})$ we define $\tau^{i_1\cdots i_r}$ as in (3.16) (with $\tilde{\mathrm{d}}x^i$ replaced by $\mathrm{d}x^i$). Then (4.1) leads to

$$\tau^{i_1\cdots i_r} = C^{i_1 i_2}{}_{k_1} C^{i_3 k_1}{}_{k_2} \cdots C^{i_r k_{r-2}}{}_{\ell}\, \mathrm{d}x^{\ell}\,. \tag{4.29}$$

Inserting this in (3.18), we obtain

$$\mathrm{d}f = D_k f\, \mathrm{d}x^k \tag{4.30}$$

with

$$D_k = \partial_k + \sum_{r=2}^{\infty} \frac{1}{r!} C^{i_1 i_2}{}_{j_1} C^{i_3 j_1}{}_{j_2} \cdots C^{i_r j_{r-2}}{}_{k}\, \partial_{i_1} \cdots \partial_{i_r}\,. \tag{4.31}$$

This is our equation (4.28).

Lemma 4.4.

$$\mathrm{d}x^i\, f = \exp\left(\mathbf{C}^k(x)\, \frac{\partial}{\partial y^k}\right)^i{}_j\, f(y)\bigg|_{y=x} \mathrm{d}x^j\,. \tag{4.32}$$

Proof. We have

$$[\mathrm{d}x^i, f] = [\mathrm{d}f, x^i] = (D_j f)\,[\mathrm{d}x^j, x^i] = (D_j f)\, C^{ij}{}_k\, \mathrm{d}x^k\,.$$

Inserting expression (4.28) for D_j, we find

$$\begin{aligned}
[\mathrm{d}x^i, f] &= \sum_{r=1}^{\infty} \frac{1}{r!} (\mathbf{C}^{k_1} \cdots \mathbf{C}^{k_{r-1}} \mathbf{C}^i)^{k_r}{}_j\, \partial_{k_1} \cdots \partial_{k_r} f\, \mathrm{d}x^j \\
&= \sum_{r=1}^{\infty} \frac{1}{r!} (\mathbf{C}^{k_1} \cdots \mathbf{C}^{k_r})^i{}_j\, \partial_{k_1} \cdots \partial_{k_r} f\, \mathrm{d}x^j \\
&= \left(\exp\left(\mathbf{C}^k(x)\, \frac{\partial}{\partial y^k}\right) - 1\right)^i{}_j f(y)\bigg|_{y=x} \mathrm{d}x^j\,.
\end{aligned}$$

Here we have stressed the possible x^k-dependence of the structure matrices which necessitates the introduction of the auxiliary variables y^k in the last expression. □

Let us suppose that $\Omega^1(\mathcal{A})$ considered as an algebra with the product $\bullet$ (see section 3) is nilpotent, i.e. there is a number $k \in \mathbb{N} \setminus \{0\}$ such that all products with k factors vanish (see [18], for example). The smallest such number is called the *index* of the algebra. Since multiplication is determined through the $n \times n$ matrices $\mathbf{C}^i$, the index can be maximally n. Then (4.28) shows that the left-partial derivatives are differential operators of at most nth order.

If $(\Omega^1(\mathcal{A}), \bullet)$ is not nilpotent, then there is a non-vanishing idempotent element (see [18], for example). The sum in (4.28) is then not finite, so that the left-partial derivatives are non-local. This is the case, for example, for the 'lattice calculus' $\boxed{\mathrm{L}}$ and for each differential calculus which is an extension in the sense of subsection 4.1.

For the *right-partial derivatives* D_{-j} defined by $\mathrm{d}f =: \mathrm{d}x^j\, D_{-j} f$, equation (4.28) is replaced for constant $\mathbf{C}$ by

$$D_{-j} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r!} (\mathbf{C}^{k_1} \cdots \mathbf{C}^{k_{r-1}})^{k_r}{}_j\, \partial_{k_1} \cdots \partial_{k_r}\,. \tag{4.33}$$

Examples. In the following we examine the four irreducible calculi which we found in subsection 4.4, and present the corresponding left- and right-partial derivatives.

(9) In this case, the only non-vanishing commutators (4.1) are

$$[dx^2, x^3] = [dx^3, x^2] = dx^1 . \tag{4.34}$$

The corresponding left- and right-partial derivatives are

$$D_{\pm 1} = \partial_1 \pm \partial_2 \partial_3 \qquad D_{\pm 2} = \partial_2 \qquad D_{\pm 3} = \partial_3 \tag{4.35}$$

in accordance with the fact that the index of the associated algebra is equal to 2. In terms of $y^1 := x^1$, $y^2 := \frac{i}{\sqrt{2}}(x^2 + x^3)$ and $y^3 := \frac{1}{\sqrt{2}}(x^2 - x^3)$ we obtain

$$[dy^\mu, y^\nu] = -\delta^{\mu\nu} dy^1 \qquad (\mu, \nu = 2, 3) . \tag{4.36}$$

This is a special case of a differential calculus which has been studied in [4, 5] (see also equation (1.3) and the second example in subsection 3.2).

(10) Here, the non-vanishing commutators are

$$[dx^2, x^3] = [dx^3, x^2] = dx^1 \qquad [dx^3, x^3] = dx^2 \tag{4.37}$$

and the left- (right-) partial derivatives are

$$D_{\pm 1} = \partial_1 \pm \partial_2 \partial_3 + \tfrac{1}{6}\partial_3^3 \qquad D_{\pm 2} = \partial_2 \pm \tfrac{1}{2}\partial_3^2 \qquad D_{\pm 3} = \partial_3 . \tag{4.38}$$

The differential of a function f thus involves third order derivatives

$$df = (\partial_1 + \partial_2\partial_3 + \tfrac{1}{6}\partial_3^3) f\, dx^1 + (\partial_2 + \tfrac{1}{2}\partial_3^2) f\, dx^2 + \partial_3 f\, dx^3 . \tag{4.39}$$

For the associated algebra the index is 3. A generalization of this new calculus to n dimensions with up to nth order left-partial derivatives will be described in the next subsection.

(11) In this case we have

$$[dx^1, x^3] = [dx^3, x^1] = dx^1 \qquad [dx^2, x^3] = [dx^3, x^2] = dx^2 \qquad [dx^3, x^3] = dx^3 \tag{4.40}$$

with the left- (right-) partial derivatives

$$D_{\pm 1} = \partial_1 \exp(\pm\partial_3) \qquad D_{\pm 2} = \partial_2 \exp(\pm\partial_3) \qquad D_{\pm 3} = \pm(\exp(\pm\partial_3) - 1) . \tag{4.41}$$

(12) Here we have the non-vanishing commutators

$$\begin{aligned} &[dx^1, x^3] = [dx^3, x^1] = dx^1 \qquad [dx^2, x^2] = dx^2 \\ &[dx^2, x^3] = [dx^3, x^2] = dx^2 \qquad [dx^3, x^3] = dx^3 \end{aligned} \tag{4.42}$$

and the left- (right-) partial derivatives

$$\begin{aligned} D_{\pm 1} &= (\partial_1 \pm \tfrac{1}{2}\partial_2^2) \exp(\pm\partial_3), \\ D_{\pm 2} &= \partial_2 \exp(\pm\partial_3) \\ D_{\pm 3} &= \pm(\exp(\pm\partial_3) - 1) . \end{aligned} \tag{4.43}$$

□

Let us consider a differential calculus which is an extension in the sense of subsection 4.1. $\tilde{D}_I$, $I = 1, \ldots, n+1$, are the corresponding left-partial derivatives and D_j, $j = 1, \ldots, n$, those of the n-dimensional calculus which generates the extension. Then we have the following result.

Lemma 4.5.

$$\check{D}_j = D_j \exp(\partial_{n+1}) \qquad (j = 1, \ldots, n) \tag{4.44}$$

$$\check{D}_{n+1} = \exp(\partial_{n+1}) - 1\,. \tag{4.45}$$

Proof. Recalling equation (4.7), we find

$$\begin{aligned} \mathrm{d}f &= [\mathrm{d}x^{n+1}, f] \\ &= \left(\exp\left(\check{\mathbf{C}}^I \frac{\partial}{\partial y^I}\right) - \mathbf{1}\right)^{n+1}_J f(y)\bigg|_{y=x} \mathrm{d}x^J \qquad \text{using (4.32)} \\ &= \left(\exp\left(\check{\mathbf{C}}^i \frac{\partial}{\partial y^i}\right) \exp\left(\frac{\partial}{\partial y^{n+1}}\right) - \mathbf{1}\right)^{n+1}_J f(y)\bigg|_{y=x} \mathrm{d}x^J \\ &= \check{D}_J f\, \mathrm{d}x^J\,. \end{aligned}$$

On functions which do not depend on x^{n+1} the $\check{D}_j$ coincide with the operators D_j. Hence

$$D_j = \left(\exp\left(\check{\mathbf{C}}^i \frac{\partial}{\partial y^i}\right) - \mathbf{1}\right)^{n+1}_j f(y)\bigg|_{y=x}\,. \tag{4.46}$$

With this observation, the conjectured formulae follow immediately. □

In the last two examples treated above, (11) and (12), we have special cases of this general result.

4.6. An n-dimensional differential calculus with up to nth-order partial derivatives

The relations

$$[\mathrm{d}x^i, x^j] = \begin{cases} \mathrm{d}x^{i+j} & \text{if } i+j \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{4.47}$$

determine a consistent differential calculus on $\mathcal{A}$. For $n = 2$ this is our calculus $\boxed{\mathbf{I}}$ and for $n = 3$ we recover the calculus (10) of subsection 4.4 (up to a renumbering of the x^i).

A partition $p(m)$ of a positive integer m is a sequence of positive integers $p_1, \ldots, p_r$ such that $\sum_{s=1}^r p_s = m$. It is always possible to write $p(m)$ in the form $(1^{k_1}, 2^{k_2}, \ldots, m^{k_m})$ where ℓ^k means that ℓ appears exactly k times in $p(m)$. With the definitions

$$p(m)! := (k_1!) \cdots (k_m!) \qquad \partial_{p(m)} := \partial_{p_1} \cdots \partial_{p_r} \tag{4.48}$$

one finds

$$\mathrm{d}f = \sum_{m=1}^{n} \sum_{p(m)} \frac{1}{p(m)!}\, \partial_{p(m)} f\, \mathrm{d}x^m\,. \tag{4.49}$$

The first four left-partial derivatives are thus

$$\begin{aligned} D_1 &= \partial_1 \\ D_2 &= \partial_2 + \tfrac{1}{2}\,\partial_1^2 \\ D_3 &= \partial_3 + \partial_2\partial_1 + \tfrac{1}{6}\,\partial_1^3 \\ D_4 &= \partial_4 + \partial_3\partial_1 + \tfrac{1}{2}\,\partial_2^2 + \tfrac{1}{2}\,\partial_2\partial_1^2 + \tfrac{1}{24}\,\partial_1^4\,. \end{aligned} \tag{4.50}$$

4.7. On some solutions of the consistency conditions

We can always decompose the structure functions $C^{ij}{}_k$ as follows

$$C^{ij}{}_k = \frac{1}{n+1}\left(\delta^i_k\, C^j + \delta^j_k\, C^i\right) + P^{ij}{}_k \tag{4.51}$$

where

$$C^i := C^{ij}{}_j \qquad P^{ij}{}_j = 0\,. \tag{4.52}$$

The first of the consistency conditions (4.4) requires that

$$P^{ij}{}_k = P^{ji}{}_k \tag{4.53}$$

and the second becomes

$$P^{ik}{}_m\, P^{jm}{}_\ell - P^{jk}{}_m\, P^{im}{}_\ell + \frac{1}{n+1}\, C^m\left(\delta^j_\ell\, P^{ik}{}_m - \delta^i_\ell\, P^{jk}{}_m\right) + \frac{1}{(n+1)^2}\, C^k\left(\delta^j_\ell\, C^i - \delta^i_\ell\, C^j\right) = 0\,. \tag{4.54}$$

For vanishing $P^{ij}{}_k$ this implies $C^i = 0$ and thus $C^{ij}{}_k = 0$. A non-commutative differential calculus therefore has to have a non-vanishing traceless part of $C^{ij}{}_k$.

In what follows, we consider some solutions of the consistency conditions for general n. With minor modifications these are taken from [19] where the consistency conditions (4.4) arose in a different context. Instead of using (4.51) it appears to be more convenient to use a corresponding decomposition with C^i replaced by vector components P^i and $P^{ij}{}_k$ not necessarily traceless. A simple solution of (4.4) is then

$$C^{ij}{}_k = \delta^i_k\, P^j + \delta^j_k\, P^i - P^i P^j U_k \tag{4.55}$$

with an additional covector† U subject to

$$U_k\, P^k = 1\,. \tag{4.56}$$

A generalization of this solution is given by

$$C^{ij}{}_k = \frac{1}{\Delta}\begin{vmatrix} \delta^i_k\, P^j_1 + \delta^j_k\, P^i_1 - P^i_1 P^j_1 U_k & \delta_{k\ell} P^\ell_1 & \delta_{k\ell} P^\ell_2 & \dots & \delta_{k\ell} P^\ell_L \\ 0 & P_1\cdot P_1 & P_1\cdot P_2 & \dots & P_1\cdot P_L \\ -(P^i_1 - P^i_2)(P^j_1 - P^j_2) & P_2\cdot P & P_2\cdot P_2 & \dots & P_2\cdot P_L \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -(P^i_1 - P^i_L)(P^j_1 - P^j_L) & P_L\cdot P & P_L\cdot P_1 & \dots & P_L\cdot P_L \end{vmatrix} \tag{4.57}$$

in terms of a determinant (cf [19]). Here $P_1, \dots, P_L$ are $L \leqslant n$ linearly independent vectors, all subject to the condition (4.56). Furthermore, we have introduced the abbreviation $P\cdot P' := \sum_{i=1}^n P^i\, P'^i$ and the subdeterminant

$$\Delta := \begin{vmatrix} P_1\cdot P_1 & P_1\cdot P_2 & \dots & P_1\cdot P_L \\ P_2\cdot P_1 & P_2\cdot P_2 & \dots & P_2\cdot P_L \\ \vdots & \vdots & \dots & \vdots \\ P_L\cdot P & P_L\cdot P_2 & \dots & P_L\cdot P_L \end{vmatrix}\,. \tag{4.58}$$

† P^i and U_j transform as the components of a vector and a covector, respectively, under $GL(n,\mathbb{C})$- or, more generally, $GL(n,\mathcal{A})$-transformations, cf subsection 4.2.

Equation (4.57) is obviously symmetric in the two indices i, j. It satisfies ${C^{ij}}_k P^k_\alpha = P^i_\alpha P^j_\alpha$, $\alpha = 1, \ldots, L$, which can then be used to prove that also the second condition in (4.4) is satisfied (cf [19])†. A different proof is given below, which, moreover, provides us with a clear characterization of the differential calculi determined by (4.57).

Lemma 4.6. Via a $GL(n, \mathcal{A})$-transformation, equation (4.57) is equivalent to

$$ {C^{ij}}_k = \sum_{\alpha=1}^{L-1} \delta^i_\alpha \delta^j_\alpha \delta^\alpha_k + \sum_{J=L}^{n-1} \delta^j_n \delta^i_J \delta^J_k + \delta^i_n \left(\delta^j_n \delta^n_k + \sum_{J=L}^{n-1} \delta^j_J \delta^J_k \right) . \tag{4.59} $$

Proof. After a cyclic renumbering of the vectors $P_1, \ldots, P_L$, (4.57) takes the form

$$ {C^{ij}}_k = \frac{1}{\Delta} \begin{vmatrix} \delta^i_k P^j_n + \delta^j_k P^i_n - P^i_n P^j_n U_k & \delta_{k\ell} P^\ell_\beta & \delta_{k\ell} P^\ell_n \\ -(P^i_n - P^i_\alpha)(P^j_n - P^j_\alpha) & P_\alpha \cdot P_\beta & P_\alpha \cdot P_n \\ 0 & P_n \cdot P_\beta & P_n \cdot P_n \end{vmatrix} $$

where $\alpha, \beta = 1, \ldots, L-1$. Now we complete the set of linearly independent vectors P_α, P_n to a linear frame (field) by adding vectors P_J, $J = L, \ldots, n-1$, such that $P_\alpha \cdot P_J = P_n \cdot P_J = 0$. Then

$$ {C^{ij}}_k P^k_\alpha = P^i_\alpha P^j_\alpha \qquad C^{ij}_k P^k_n = P^i_n P^j_n \qquad {C^{ij}}_k P^k_J = P^{(i}_n P^{j)}_J - P^i_n P^j_n V_J $$

where $V_J := U_k P^k_J$. Let $P \in GL(n, \mathcal{A})$ be the matrix with entries P^i_j. The transformation $x'^k = (P^{-1})^k_\ell x^\ell$ preserves (4.1) if $\mathbf{C}'^i = (P^{-1})^i_j (P^{-1} \mathbf{C}^j P)$. Here it leads to

$$ {C'^{ij}}_k = \sum_\alpha \delta^i_\alpha \delta^j_\alpha \delta^\alpha_k + \sum_J \delta^{(i}_n \delta^{j)}_J \delta^J_k + \delta^i_n \delta^j_n \left(\delta^n_k - \sum_J V_J \delta^J_k \right) $$

so that

$$ \mathbf{C}^\alpha = \left(\begin{array}{c|c|c} E^r & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \qquad \mathbf{C}^J = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & e^J & 0 \end{array} \right) \qquad \mathbf{C}^n = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & -V & 1 \end{array} \right) $$

where $(E^\alpha)^\beta_\gamma = \delta^\beta_\alpha \delta^\alpha_\gamma$ and $(e^J)_K = \delta^J_K$. I is the $(n-L) \times (n-L)$ unit matrix. A further $GL(n, \mathcal{A})$-transformation with

$$ A = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & V & 1 \end{array} \right) $$

eliminates the V-term. □

According to lemma 4.6, the calculus determined by (4.57) is $\hat\otimes^n \boxed{\mathbf{L}}$ for $L = n$ and $(\hat\otimes^{L-1} \boxed{\mathbf{L}}) \,\hat\otimes\, \mathbf{Ext}(\hat\otimes^{n-L} \boxed{\mathbf{K}})$ for $L < n$ where **Ext** indicates an extension in the sense of subsection 4.1. Comparison with the list of calculi in subsection 4.4 shows that the *ansatz (4.57) does not exhaust the possibilities by far.*

One can try other ansätze, but it is unlikely that this can be made into a systematic procedure to obtain the complete set of solutions of the consistency conditions (4.4).

† We were unable to verify the statement in [19] that certain linear combinations of terms of the form (4.57) (cf equation (28) of [19]) also satisfy the nonlinear condition in (4.4). A counter-example is given by $n = 3, L = 2$ with $P^i_1 = \delta^i_1$, $P^i_2 = \delta^i_2$ (and $U_k = \delta^1_k + \delta^2_k$).

5. Conclusions

In this work we have started a systematic exploration of differential calculi on commutative algebras and presented several new results.

A central part of this work is the classification of three-dimensional differential calculi with constant structure functions (on a commutative algebra with three generators). Much of the additional material in this paper provides the necessary background or arose from insights obtained via this classification. Apart from having solved the classification problem for $n \leqslant 3$, we have presented generalizations to arbitrary dimension n for all the calculi found in this way.

Only four of the three-dimensional calculi (more precisely: $GL(3, \mathbb{C})$ orbits) obtained in subsection 4.4 turned out to be irreducible, i.e. they are not skew tensor products of lower-dimensional calculi. Two of these are extensions of two-dimensional calculi (in the sense of subsection 4.1). They are 'non-local' in the sense that their left- (or right-) partial derivatives involve finite difference operators. The remaining two genuinely three-dimensional calculi have local (though higher-order) left- and right-partial derivatives. For one of them, a generalization to arbitrary dimension is already known [4, 5]. A corresponding generalization of the other calculus is presented in subsection 4.6, the left-partial derivatives are differential operators of up to nth order.

Our classification procedure extends to $n > 3$, but the corresponding calculations become much more involved. Computer algebra should then be helpful.

For the new calculi we have so far been unable to find any relationsship with structures of interest in other branches of mathematics or in physics, similar to what we have for the examples mentioned in the introduction. Further investigation of these calculi is therefore required. There is, however, a general aspect which supports our investigation from a physical point of view. A study of differential calculi on finite sets has shown that the choice of a differential calculus assigns to a set a structure which should be regarded as an analogue of that of a differentiable manifold [6]. The present paper provides new examples of such generalized spaces which can be regarded as deformations of $\mathbb{R}^n$ with the ordinary differential calculus. Such a deformation induces (in a universal manner) corresponding deformations of models and theories built on the differential calculus (cf [3, 4, 6]). There is the hope to obtain in this systematic way physical models which are somehow close to known models but which improve the latter by properties like complete integrability or finiteness (of quantum perturbation theory, in particular for non-renormalizable theories like gravity).

References

[1] Hermann R 1977 *Quantum and Fermion Differential Geometry, Part A (Interdisciplinary Mathematics XVI)* (Brookline, MA: Math. Sci. Press)

[2] Connes A and Lott J 1990 Particle models and noncommutative geometry *Nucl. Phys. (Proc. Suppl.)* B **18** 29

[3] Dimakis A, Müller-Hoissen F and Striker T 1993 Non-commutative differential calculus and lattice gauge theory *J. Phys. A: Math. Gen.* **26** 1927

[4] Dimakis A and Müller-Hoissen F 1993 A noncommutative differential calculus and its relation to gauge theory and gravitation *Int. J. Mod. Phys. (Proc. Suppl.)* A **3** 474; 1992 Noncommutative differential calculus, gauge theory and gravitation *Preprint* Göttingen University GOE-TP 33/92

[5] Dimakis A and Müller-Hoissen F 1993 Stochastic differential calculus, the Moyal $*$-product, and noncommutative geometry *Lett. Math. Phys.* **28** 123; 1994 Noncommutative differential calculus: quantum groups, stochastic processes, and the antibracket *Adv. Applied Clifford Algebras (Proc. Suppl.)* **4** (S1) 113

[6] Dimakis A and Müller-Hoissen F 1994 Differential calculus and gauge theory on finite sets *J. Phys. A: Math. Gen.* **27** 3159; 1994 Discrete differential calculus, graphs, topologies and gauge theory *J. Math. Phys.* **35** 6703

Dimakis A, Müller-Hoissen F and Vanderseypen F 1994 Discrete differential manifolds and dynamics on networks *Preprint* hep-th/9408114 (*J. Math. Phys.* to appear)

[7] Borowiec A, Kharchenko V K and Oziewicz Z 1994 On free differentials on associative algebras *Non Associative Algebra and its Applications* ed S Gonzales (Dordrecht: Kluwer); 1993 Differentials with uniqueness property *Preprint*

[8] Dimakis A and Müller-Hoissen F 1992 Quantum mechanics as non-commutative symplectic geometry *J. Phys. A: Math. Gen.* **25** 5625

[9] Baehr H C 1994 Differentialkalküle über kommutativen Algebren *Diploma thesis* Göttingen University (in German)

[10] Bourbaki N 1989 *Elements of Mathematics, Algebra I* (Berlin: Springer) chs 1–3

[11] Kastler D 1988 *Cyclic cohomology within the differential envelope* (Paris: Hermann)

[12] Greub W H 1967 *Multilinear Algebra* (Berlin: Springer)

[13] Kunz E 1986 *Kähler Differentials* (Braunschweig: Vieweg)

[14] Müller-Hoissen F and Reuten C 1993 Bicovariant differential calculus on $GL_{p,q}(2)$ and quantum subgroups *J. Phys. A: Math. Gen.* **26** 2955

[15] Bresser K 1995 All bicovariant differential calculi on $GL_q(3, \mathbb{C})$ and $SL_q(3, \mathbb{C})$ *J. Phys. A: Math. Gen.* **28** 2545

[16] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[17] Doplicher S, Fredenhagen K and Roberts J E 1994 Spacetime quantization induced by classical gravity *Phys. Lett.* **331B** 39

[18] Abian A 1971 *Linear Associative Algebras* (Oxford: Pergamon)

[19] Bouguenaya Y and Fairlie D B 1986 A finite difference scheme with a Leibniz rule *J. Phys. A: Math. Gen.* **19** 1049